Output Feedback Stabilization and Optimization of Multivariable Dynamical Systems using Genetic Algorithms

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ABSTRACT- The computation of the optimal constant output feedback gains for linear multivariable systems with respect to a quadratic performance criterion is discussed. The purpose of this paper is to determine an appropriate feedback matrix and to stabilize the system by minimizing a quadratic cost using Genetic Algorithm.

Keywords: Optimal control, stabilization, output feedback, Genetic Algorithms, Optimization.

I. INTRODUCTION

The static output feedback control problem received a lot of attention in the control literature. Although the fundamental question of existence of stabilizing output feedback controllers is still open, a tremendous effort has been invested towards developing computational approaches to solve this problem if a solution exists. One of the popular design approaches of static output feedback controllers is the solution of a linear quadratic (LQ) optimization problem analogously to the case of full state feedback.

There are basically two computational approaches, both iterative, for the solution of the optimal LQ output feedback problem. The first approach, pioneered for continuous-time systems by Levine and Athans (1970)[1], where for the computation of the optimal feedback matrix, the necessary conditions for the problem lead to a set of coupled non linear matrix equations, problems that are characterized by the presence of slow and fast modes give rise to ill conditioning in the dynamic.

The other approach has been presented by Anthony J. Calise and Daniel D. Moerder (1985)[2]. A design procedure for stabilizing a system by minimizing a quadratic cost was proposed. It requires only the solution of linear matrix equations. While this method presents some problems essentially related to the initializing of the algorithm.

Hence, to overcome the complexity of the design problem, we present in this paper another numerical solution of the optimal LQ output feedback control problem using the optimization ability of genetic algorithms, where the standard infinite quadratic performance criterion is considered as fitness function. Genetic Algorithms (GA), are powerful optimization methods which can be applied in order to find the best controller parameters. They are substantially “trial and error” methods based on the way biological evolution works.

The rest of this paper is organized as follows. In section 2 we present the optimal output feedback control formulation. In section 3, we describe the approach adopted in [2] for the determining of the output feedback gain of linear time-invariant systems. In section 4 we propose the resolution of the problem using GA. The effectiveness of the GA optimization method is illustrated in section 5 through a numerical example. The conclusion is presented in section 6.

II- OPTIMAL OUTPUT CONTROL FORMULATION

The basic problem of output feedback control of linear time-invariant systems lies in selecting the appropriate gain matrix.
The first step in the development of this design procedure is the formulation and then the proposed solution of the mathematical optimization problem. This optimization problem begins with a time invariant system of the form:

\[ \dot{x}(t) = Ax(t) + Bu(t). \]  
\[ y(t) = Cx(t). \]

Where the state vector \( x(t) \) is an \( n \) vector, the control vector \( u(t) \) is \( m \) dimensional and the output \( y(t) \) is \( p \) dimensional. \( A, B \) and \( C \) are real matrices.

As a first try a reasonable performance measure, begin with the standard quadratic performance criterion.

\[ J_0 = \frac{1}{2} \int_0^\infty \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] dt. \]  

Where \( Q \geq 0 \) and \( R > 0 \).

(1) and (3) form an optimization problem for which the optimal control can be generated by \( u^*(t) = -Gx(t) \). The feedback gain matrix \( G \) can be evaluated through the solution of an algebraic Riccati equation.

Our goal now is to design a static output feedback control law with time invariant feedback gain which minimizes cost index \( J_0 \).

\[ u(t) = -Fx(t) \]  

Where \( F \) is an \( (m \times p) \) feedback gain matrix designed for the system in (1), to be determined. Substituting (4) into (1), the closed loop dynamics for the system become:

\[ \dot{x} = \hat{A}x, \quad x(0) = x_0 \]  

where \( \hat{A} = A - BFC \)  

It should be clear that \( F \) is regarded as the control for the system (5).

The necessary and sufficient conditions such that the system (1) is asymptotically stable are:

\[ \text{Re} \lambda_i(\hat{A}) < 0 \]  

Where we design by \( \text{Re} \lambda_i(\hat{A}) \) the real part of the \( i \)th eigenvalue of \( \hat{A}, \ i = 1, \ldots \ n \)

We assume here that there exists an \( F \) such that the previous condition can be met and methods for determining \( F \) must be fixed then.

### III- Problem Resolution Using a Constrained Static Optimization Problem.

We present first in this section the approach presented in [2], for the determining of the output feedback gain of systems with ill-conditioned dynamics. This approach is applied in this section to linear time-invariant systems.

We consider the system given in (1), from (2) and (4) the control can be written as

\[ u = -FCx \]  

In order to obtain necessary conditions for determining \( F \) the performance in (3) can be expressed as

\[ J_0 = \text{tr}\{Kx_0x_0^T\} \]  

Where the constrained dynamic optimization problem is converted into a constrained static optimization problem where \( K \) satisfies the matrix Lyapunov equation

\[ S(F,K) = \hat{A}^T K + K\hat{A} + \mathcal{R} = 0 \]  

\[ \mathcal{R} = Q + C^T F^T RFC \]  

Where the matrix \( \hat{A} \) supposed stable, characterizes the closed loop system.

The dependency of the optimal solution on the initial condition \( x_0 \) can be removed if we consider the

\[ E(x_0x_0^T) = I_n \]  

To obtain necessary conditions for minimizing \( J_0 \) with respect to \( F \) and \( K \) subject to the constrain in (10), we can apply gradient matrix operations to the Lagrangian

\[ L(F,K) = \text{tr}\{K\} + \text{tr}\{S(F,K)L_0^T\} \]  

Where \( L_0 \) is Lagrange multiplier.

We minimize \( L(F,K) \), The necessary conditions follow from:

\[ \frac{\partial L}{\partial F} = \frac{\partial L}{\partial K} = \frac{\partial L}{\partial L_0} = 0 \]  

The necessary conditions for the optimal output feedback can be expressed as:

\[ F = R^{-1}(B^T KLC^T)(C L_0C^T)^{-1} \]  

\[ T(F,L_0) = \hat{A}L_0 + L_0\hat{A}^T + I_n = 0 \]
\[ S(F,K) = \dot{A}^T K + K\dot{A} + 9\mathbf{R} = 0 \quad (17) \]

A computational procedure proposed in [2], for finding an \( F \) that satisfies (15) is given in terms of the following steps:

1: choose an initial matrix \( F^0 \) for which \( \hat{A}_0 = A - BF^0C \) is stable. Set \( i=0 \).
2: Solve Lyapunov equations (16) and (17) for \( K^i, L_0^i \) functions of \( F^i \).
3: Solve from (15), for the gradient
\[ \Delta F^i = R^{-1}B^T K^i L_0^i C^T (CL_0 C^T)^{-1} - F^i \quad (18) \]
4: Choose \( \alpha^i \in [0,1] \) so that
\[ J(F^i + \alpha^i \Delta F^i) < J(F^i) = tr\{K^i\} \quad (19) \]
and \( A - B(F^i + \alpha^i \Delta F^i)C \) is stable.
5: set \( i = i + 1 \) and go to 2, with \( F^{i+1} = F^i + \alpha^i \Delta F^i \quad (20) \)

As it’s shown above, a major difficulty in designing output feedback systems is the numerical problem associated with the computation of the optimal feedback gain matrix.

The second problem consists on how to choose for each iteration \( i \), the parameter \( \alpha^i \) relative to the 4th step of the algorithm, so that \( F^i + \alpha^i \Delta F^i \) is stabilizing.

In the next paragraph, in order to determine the optimal feedback parameter of the controller, another approach is presented, a genetic algorithm is carefully designed.

VI- PROBLEM RESOLUTION USING GENETIC ALGORITHM.

The fundamental concept of Genetic Algorithm is to perform a systematic random search for the fittest individual through a number of generations of a population. The design variables are represented as strings on binary variables that corresponding to the chromosomes in natural genetics. The objective function value corresponding to design variables plays the role of fitness in natural genetics. A GA approach requires a population of chromosomes representing a combination of features from a set of features and a cost function (fitness function). First, the algorithm creates an initial population of size, which is created from a random selection of the parameters. Each parameter set represents the individual’s chromosomes. Each of the individuals is assigned a fitness based on how well each individual’s chromosomes allow it to perform in its environment. There are then three operations which occur in GAs to create the next generation: reproduction, crossover, and mutation. The process of mating and child creation is continued until an entirely new population of size is generated with the hope that strong parents will create a fitter generation of children and until a specified criterion is achieved.

GAs offer a generational improvement in the fitness of the chromosomes and after many generations will create chromosomes containing the optimized variable setting.

The application of a Genetic Algorithm is depicted in block diagram form in figure 1:

![Fig. 1. Block diagram illustrating application of Genetic Algorithm.](image-url)
The goal for the particular problem described here in, is to determine the output feedback control matrix gain $F$ with respect to a given quadratic performance criterion $J$ using Genetic Algorithm.

The proposed method:
- Uses the design of the full order system,
- Is flexible and allows some kind of constraints on the signal,
- Doesn't need a lot of time of study,

To apply the proposed method, it is necessary to have a genetic algorithm based program minimizing a fitness function.

We consider the system given in (1), the performance criterion given in (9) and the hypothesis considered in (12). The fitness function becomes.

$$J_0 = tr\{K\}$$ (21)

The fitness function numerically encodes the performance of the chromosome and can be any non linear, non differentiable, discontinuous, because the algorithm only needs a fitness assigned to each string.

All steps considered with the algebraic method are also retreated here. The computational procedure proposed in [2] for finding $F$ is then replaced by an optimization problem using genetic algorithm.

The formulation of this optimization problem may be stated as follows:

Find $F$ to minimize $J_0$

with $F_l \leq F \leq F_u$.

$J_0$ is the objective function, $F$ is the design variable matrix; the subscripts $l$ and $u$ represents lower and upper limits on design variables, respectively.

To maximize a fitness function in GA, the pseudo-objective function is transformed as follows.

$$J = cte - J_0$$

where the $cte$ is chosen to be greater than the largest value of pseudo-objective function.

One constraint is to ensure that the output of the full system is observable.

In fact, if a particular value of $F$ leads to an unstable system, the output will takes big values that leads also to great value of the fitness that will be directly eliminated through natural selection so that the individual representing $F$ will also be eliminated, stipulate that only best chromosomes with best fitness are kept for the next population: the principle of survival of the fittest.

To check for the stability of the closed loop system, we can verify that $A - BF_{opt}C$ is stable, i.e.

$$\Re\lambda(A - BF_{opt}C) < 0$$

A numerical example is presented, in the next paragraph to illustrate the effectiveness of the proposed approach.

V- ILLUSTRATIVE EXAMPLE.

Consider the following linear time-invariant system

$$\dot{x}(t) = (A - BFC)x(t)$$

Where

$$A = \begin{bmatrix} 1 & -1 & -3 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system is controllable and observable.

Consider also the performance criterion given in (21)

$$J_0 = tr\{K\}$$

Using genetic algorithm we have to determine the output feedback controller gain $F$ in order to minimise the fitness function $J_0$ in the present case study.

In implementation, we represent each variable by a 32-bit binary string chromosome, and variables are searched in $[-1, 6]$ interval. One point crossover and one bit mutation are applied with rates of 0.7 and 0.04 respectively. Size population considered is 400.

This leads to the following parameters of the feedback matrix gain controller:

$$F_{opt} = \begin{bmatrix} 2.9732 & -0.0357 \\ 0.3508 & 5.5283 \end{bmatrix},$$

and the fitness function $J_0 = 10.84$
For the iterative method proposed in [2], the initial solution is:

\[
F_0 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix},
\]

After N=24 iterations, the Algorithm converge and we obtain appropriate values of the feedback gain matrix:

\[
F_{opt} = \begin{bmatrix} 2.9655 & -0.0094 \\ 0.2759 & 5.4870 \end{bmatrix} \text{ with } J_0 = 10.84.
\]

In figures (2), (3), (4), (5) and (6) we have presented respectively the evolution of the state variables \(x_1(t)\), \(x_2(t)\), \(x_3(t)\), \(x_4(t)\), \(x_5(t)\). The time histories of the control \(u(t)\) are reported in figures (7) and (8) respectively. All figures illustrate the comparison between results obtained from Athans and al’s approach (theoretical approach) and GA approach.
Simulation results indicate that the GA, using the quadratic performance criterion as fitness function, overcomes many of the difficulties associated with existing tuning rules, and leads to satisfactory results.

The iterative method presents some problems essentially related to the initializing of the algorithm, it necessitates an initial stabilizing matrix gain, and can converge toward a local extremum, the optimum depends on the initial solution.

In the application shown in this paper, the GA was able to converge toward the convenient value of the control law parameter.

Fundamentally, GA technique is independent of the process model and consequently can be used to obtain consistent performance for a wide variety of process model. In fact, this method can be also used for non linear systems.

These results will be improved if we minimize a general criterion, adopted as fitness function, containing time domain constrained related to the overshoot, the rise time and the static error [8].

### VI- CONCLUSION

In this work we have proposed a new numerical method for the resolution of the optimal LQ output feedback control problem. This method used the optimization ability of genetic algorithms and the inherent advantages of such a technique, where the standard infinite quadratic performance criterion is considered as fitness function.

Figures depicted that both feedback matrix gain obtained with GA approach and the one obtained with the approach presented in [2], stabilizes the system and leads to good performances.

It’s clear, referring to simulation results, that the two control laws have high performances and lead to very close responses. This indicate that both methods converge to the same optimal solution.

Furthermore, with both approaches represented by the GA and the recursive method, the state variable evolutions are superposed which highlight the efficiency of the proposed GA approach compared to the other one.

### 6- REFERENCES


